Gupta et al. *Journal of Inequalities and Applications* 2012, **2012**:298
http://www.journalofinequalitiesandapplications.com/content/2012/1/298

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Duality for nondifferentiable multiobjective higher-order symmetric programs over cones involving generalized (F, α, ρ, d)-convexity

SK Gupta[1*], N Kailey[2] and Sumit Kumar[3]

*Correspondence:
skgiitr@gmail.com
[1]Department of Mathematics, Indian Institute of Technology, Patna, 800 013, India
Full list of author information is available at the end of the article

Abstract
In this paper, a pair of Wolfe type higher-order symmetric nondifferentiable multiobjective programs over arbitrary cones is formulated and appropriate duality relations are then established under higher-order-K-(F, α, ρ, d)-convexity assumptions. A numerical example which is higher-order K-(F, α, ρ, d)-convex but not higher-order K-F-convex has also been illustrated. Special cases are also discussed to show that this paper extends some of the known works that have appeared in the literature.
MSC: 90C29; 90C30; 49N15

Keywords: higher-order symmetric duality; support function; multiobjective programming; cones; efficient solutions

1 Introduction

Mangasarian [1] introduced the concept of second- and higher-order duality for nonlinear problems. He has also indicated that the study is significant due to the computational advantage over the first-order duality as it provides tighter bounds for the value of the objective function when approximations are used. Motivated by the concept in [1], several researchers [2–19] have worked in this field.

Multiobjective optimization has a large number of applications. As an example, it is generally used in goal programming, risk programming *etc.* Optimality conditions for multiobjective programming problems can be found in Miettinen [20] and Pardalos *et al.* [21]. Recently, Chinchuluun and Pardalos [22] discussed recent developments in multiobjective optimization. These include optimality conditions, applications, global optimization techniques, the new concept of epsilon pareto optimal solutions and heuristics.

Chen [7] considered a pair of symmetric higher-order Mond-Weir type nondifferentiable multiobjective programming problems and established usual duality results under higher-order F-convexity assumptions. Gulati and Gupta [9] proved duality theorems for a pair of Wolfe type higher-order nondifferentiable symmetric dual programs. Ahmad *et al.* [6] formulated a general Mond-Weir type higher-order dual for a nondifferentiable multiobjective programming problem and established usual duality results.

Gulati and Geeta [23] pointed out certain omissions in some papers on symmetric duality in multiobjective programming and discussed their corrective measures. Later on, Ahmad and Husain [5] and Gulati *et al.* [11] formulated second-order multiobjective symmetric dual programs with cone constraints and established duality results under second-order invexity assumptions. An omission in the strong duality theorem in Yang *et al.* [17] has been rectified in Gupta and Kailey [12].

The concept of (F,ρ)-convexity was introduced by Preda [24] as an extension of F-convexity [25] and ρ-convexity [26]. Yang *et al.* [18] formulated several second-order duals for scalar programming problem and proved duality results involving generalized F-convex functions. Zhang and Mond [19] extended the class of (F,ρ)-convex functions to second-order and obtained duality relations for multiobjective dual problems.

Motivated by various concepts of generalized convexity, Liang *et al.* [27] introduced a unified formulation of generalized convexity, called (F,α,ρ,d)-convexity and obtained corresponding optimality conditions and duality relations for a single objective fractional problem. This was later on extended to multiobjective fractional programming problem in Liang *et al.* [28]. Inspired by the concept given in [19, 27, 28], Ahmad and Husain [29] introduced second-order (F,α,ρ,d)-convex functions and proved duality relations for Mond-Weir type second-order multiobjective problems. In the recent work of Ahmad and Husain [30], an attempt is made to remove certain omissions and inconsistencies in the work of Mishra and Lai [31].

Agarwal *et al.* [2] achieved duality results for a pair of Mond-Weir type multiobjective higher-order symmetric dual programs over arbitrary cones under higher-order K-F-convexity assumptions. Recently, Agarwal *et al.* [32] have filled some gap in the work of Chen [7] and proved a strong duality theorem for Mond-Weir type multiobjective higher-order nondifferentiable symmetric dual programs.

In this paper, we formulate a pair of symmetric higher-order Wolfe type nondifferentiable multiobjective programs over arbitrary cones and prove weak, strong and converse duality theorems under higher-order-K-(F,α,ρ,d)-convexity assumptions. We also give a nontrivial example of a function lying in the class of higher-order K-(F,α,ρ,d)-convex but not in the class of higher-order K-F-convex. Our study extends some of the known results that appeared in [4, 8, 16, 17].

2 Notations and preliminaries

Consider the following multiobjective programming problem:

$$\begin{aligned} &K\text{-minimize} \quad \phi(x) \\ &\text{subject to} \quad x \in X^0 = \{x \in S : -g(x) \in C\}, \end{aligned} \tag{P}$$

where $S \subseteq R^n$ is open, $\phi : S \to R^k$, $g : S \to R^m$, K is a closed convex pointed cone in R^k with $\operatorname{int} K \neq \phi$ and C is a closed convex cone in R^m with nonempty interior.

Definition 1 [33] The positive polar cone C^* of C is defined as

$$C^* = \{z : \xi^T z \geqq 0, \text{ for all } \xi \in C\}.$$

Definition 2 [33] A point $\bar{x} \in X^0$ is a weak efficient solution of (P) if there exists no other $x \in X^0$ such that

$$\phi(\bar{x}) - \phi(x) \in \operatorname{int} K.$$

Definition 3 [34] A point $\bar{x} \in X^0$ is an efficient solution of (P) if there exists no other $x \in X^0$ such that

$$\phi(\bar{x}) - \phi(x) \in K \setminus \{0\}.$$

Definition 4 [2, 9, 16] Let D be a compact convex set in R^n. The support function of D is defined by

$$S(x \mid D) = \max\{x^T y : y \in D\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that

$$S(y \mid D) \geqq S(x \mid D) + z^T(y - x) \quad \text{for all } y \in D.$$

The subdifferential of $S(x \mid D)$ is given by

$$\partial S(x \mid D) = \{z \in D : z^T x = S(x \mid D)\}.$$

For any set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0 \text{ for all } z \in S\}.$$

It can be easily seen that for a compact convex set D, y is in $N_D(x)$ if and only if $S(y \mid D) = x^T y$, or equivalently, x is in $\partial S(y \mid D)$.

Definition 5 [24, 28] A functional $F : X \times X \times R^n \mapsto R$ (where $X \subseteq R^n$) is sublinear with respect to the third variable if, for all $(x, u) \in X \times X$,

(i) $F(x, u; a_1 + a_2) \leqq F(x, u; a_1) + F(x, u; a_2)$ for all $a_1, a_2 \in R^n$, and
(ii) $F(x, u; \alpha a) = \alpha F(x, u; a)$, for all $\alpha \in R_+$ and $a \in R^n$.

For notational convenience, we write

$$F(x, u; a) = F_{x,u}(a).$$

Let $F : S \times S \times R^n \mapsto R$ be a sublinear functional with respect to the third variable. Now, we recall the concept of higher-order K-F-convexity introduced in [2].

Definition 6 A differentiable function $\phi : S \to R^k$ is said to be higher-order K-F-convex at u on S with respect to $\zeta : S \times R^n \mapsto R^k$ if, for all $x \in S$ and $q \in R^n$, we have

$$\begin{aligned}&\{\phi_1(x) - \phi_1(u) - \zeta_1(u, q) + q^T \nabla_q \zeta_1(u, q) - F_{x,u}[\nabla_x \phi_1(u) + \nabla_q \zeta_1(u, q)], \dots, \\ &\quad \phi_k(x) - \phi_k(u) - \zeta_k(u, q) + q^T \nabla_q \zeta_k(u, q) - F_{x,u}[\nabla_x \phi_k(u) + \nabla_q \zeta_k(u, q)]\} \in K.\end{aligned}$$

Remark 1

(i) If $K = R^k_+$, then the above definition reduces to higher-order F-convexity given in [7, 9, 32].
(ii) If $K = R^k_+$ and $\zeta_i(u, q) = \frac{1}{2} q^T \nabla_{xx} \phi_i(u) q$, $i = 1, 2, \dots, k$, then Definition 6 becomes second-order F-convexity as considered in [12, 17].

(iii) Taking $K = R_+^k$, $\zeta_i(u,q) = \frac{1}{2}q^T\nabla_{xx}\phi_i(u)q$ and $F_{x,u}(a) = \eta(x,u)^T a$, then the definition reduces to second-order invexity (or η bonvexity) given in [5, 11, 15].

Definition 7 A differentiable function $\phi : S \to R^k$ is said to be higher-order K-(F,α,ρ,d)-convex at u on S with respect to $\zeta : S \times R^n \mapsto R^k$ if for all $x \in S$ and $q \in R^n$, there exist vector $\rho \in R^k$, a real valued function $\alpha : S \times S \to R_+\backslash\{0\}$ and $d : S \times S \to R^k$ such that

$$\begin{aligned}&\bigl\{\phi_1(x) - \phi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q) - F_{x,u}\bigl[\alpha(x,u)\bigl(\nabla_x\phi_1(u) + \nabla_q\zeta_1(u,q)\bigr)\bigr]\\&\quad - \rho_1 d_1^2(x,u),\ldots,\phi_k(x) - \phi_k(u) - \zeta_k(u,q) + q^T\nabla_q\zeta_k(u,q)\\&\quad - F_{x,u}\bigl[\alpha(x,u)\bigl(\nabla_x\phi_k(u) + \nabla_q\zeta_k(u,q)\bigr)\bigr] - \rho_k d_k^2(x,u)\bigr\} \in K.\end{aligned}$$

Remark 2

(i) If $K = R_+$, $\alpha(x,u) = 1$ and $\zeta_i(u,q) = \frac{1}{2}q^T\nabla_{xx}\phi_i(u)q$, $i = 1,2,\ldots,k$, the definition of higher-order K-(F,α,ρ,d)-convexity reduces to second-order (F,ρ)-convexity given by Srivastava and Bhatia [14].

(ii) If $K = R_+$, $\alpha(x,u) = 1$ and $\rho = 0$, then Definition 7 reduces to higher-order F-convexity (see [7, 9]).

(iii) If we take $K = R_+^1$, $\alpha(x,u) = 1$, $\rho = 0$, $\zeta_1(u,q) = \frac{1}{2}q^T\nabla_{xx}\phi_1(u)q$ and $F_{x,u}(a) = \eta(x,u)^T a$, where η is a function from $S \times S$ to R^n, then the above definition becomes second-order η-convexity given in [15].

Next, we illustrate a nontrivial example of higher-order K-(F,α,ρ,d)-convex functions which are not higher-order K-F-convex.

Example 1 Let $X = [-2.5,-0.5] \subset R$, $n = m = 1$, $k = 2$, $K = \{(x,y) : x \geqq 0, y \geqq 0\}$. Consider the function $\psi : X \to R^2$ to be defined by $\psi(x) = (\psi_1(x), \psi_2(x))$, where

$$\psi_1(x) = x^3\sin\frac{2}{x}, \qquad \psi_2(x) = x^3,$$

and $\alpha : X \times X \to R_+\backslash\{0\}$ to be identified by $\alpha(x,u) = (u^2+1)$. Let the functional $F : X \times X \times R \to R$ be defined by

$$F_{x,u}(a) = 12(1-u)a.$$

Suppose $d : X \times X \to R^2$ is given by $d(x,u) = (d_1(x,u), d_2(x,u))$, where

$$d_1(x,u) = \bigl(x^4 + u^2\bigr)^{\frac{3}{2}}, \qquad d_2(x,u) = \bigl(x^2 + u^2\bigr)^{\frac{5}{2}}$$

and $\zeta : X \times R \to R^2$ is defined by $\zeta(u,q) = (\zeta_1(u,q), \zeta_2(u,q))$, where

$$\zeta_1(u,q) = 15u + 12q, \qquad \zeta_2(u,q) = \sin^2 u + q^2.$$

To show ψ is higher-order (F,α,ρ,d)-convex, we need to prove

$$\begin{aligned}L = {}&\bigl\{\psi_1(x) - \psi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q)\\&- F_{x,u}\bigl[\alpha(x,u)\bigl(\nabla_x\psi_1(u) + \nabla_q\zeta_1(u,q)\bigr)\bigr]\end{aligned}$$

$$- \rho_1 d_1^2(x,u), \psi_2(x) - \psi_2(u) - \zeta_2(u,q) + q^T \nabla_q \zeta_2(u,q)$$
$$- F_{x,u}\big[\alpha(x,u)\big(\nabla_x \psi_2(u) + \nabla_q \zeta_2(u,q)\big)\big] - \rho_2 d_2^2(x,u)\big\} \in K,$$

which for $\rho_1 = -4$ and $\rho_2 = -28$ gives

$$\begin{aligned} L &= \bigg\{x^3 \sin\frac{2}{x} - u^3 \sin\frac{2}{u} - 15u - 12(1-u)\bigg[\big(u^2+1\big)\bigg(-2u\cos\frac{2}{u} + 3u^2 \sin\frac{2}{u} + 12\bigg)\bigg] \\ &\quad + 4\big(x^4+u^2\big)^3, x^3 - u^3 - \sin^2 u + q^2 - 12(1-u)\big[\big(u^2+1\big)\big(3u^2+2q\big)\big] \\ &\quad + 28\big(x^2+u^2\big)^5\bigg\} \in K \\ &= \{L_1, L_2\} \in K, \end{aligned}$$

where

$$\begin{aligned} L_1 &= x^3 \sin\frac{2}{x} - u^3 \sin\frac{2}{u} - 15u - 12(1-u)\bigg[\big(u^2+1\big)\bigg(-2u\cos\frac{2}{u} + 3u^2 \sin\frac{2}{u} + 12\bigg)\bigg] \\ &\quad + 4\big(x^4+u^2\big)^3 \\ &\geqq 0 \quad \forall x, u \in X \text{ as can be seen from Figure 1} \end{aligned} \tag{E1}$$

and

$$\begin{aligned} L_2 &= x^3 - u^3 - \sin^2 u + q^2 - 12(1-u)\big[\big(u^2+1\big)\big(3u^2+2q\big)\big] + 28\big(x^2+u^2\big)^5 \\ &= L_{21} + L_{22}. \end{aligned}$$

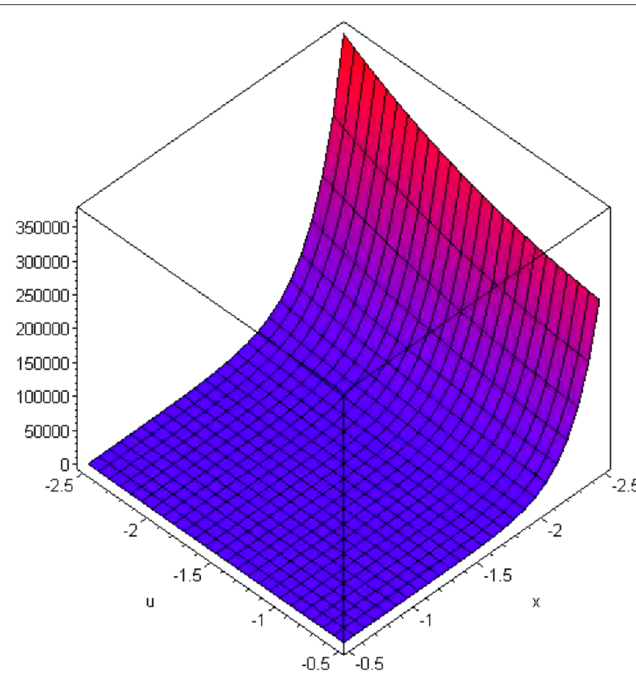

Figure 1 Graph of L_1 (Expression (E1)) against x and u.

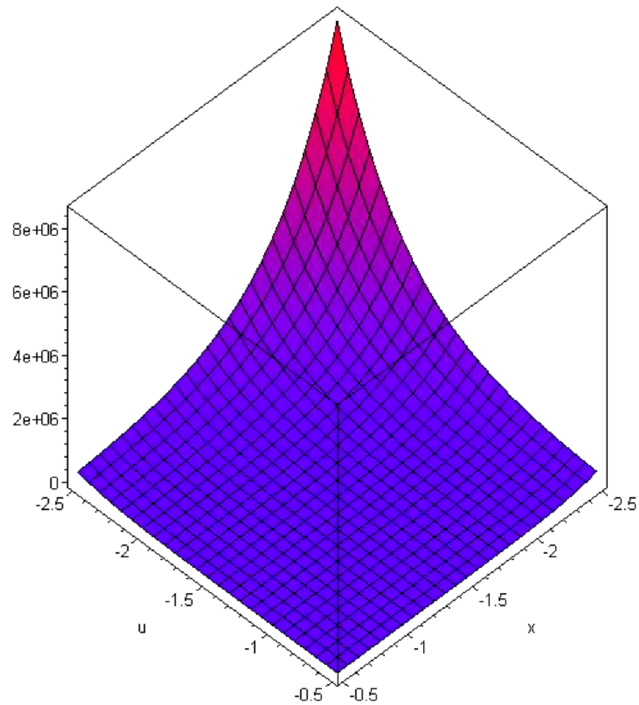

Figure 2 Graph of L_{21} (Expression (E2)) against *x* and *u*.

But

$$\begin{aligned} L_{21} &= x^3 - u^3 - \sin^2 u - 12(1-u)\left[3u^2\left(u^2+1\right)\right] + 28\left(x^2+u^2\right)^5 \\ &\geqq 0 \quad \forall x, u \in X \text{ as can be seen from Figure 2} \end{aligned} \tag{E2}$$

and

$$\begin{aligned} L_{22} &= q^2 - 12(1-u)\left[2q\left(u^2+1\right)\right] \\ &\geqq 0 \quad \forall u \in X \text{ and } q \in \left(-10^{18}, 10^{18}\right) \text{ as can be seen from Figure 3.} \end{aligned} \tag{E3}$$

Hence, $L_2 \geqq 0$. Therefore, ψ is higher-order K-(F, α, ρ, d)-convex with respect to ζ.

Next, we need to show that ψ is not higher-order K-F-convex with respect to ζ. To prove it, we will show that

$$\begin{aligned} M = \{&\psi_1(x) - \psi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q) - F_{x,u}\left[\nabla_x\psi_1(u) + \nabla_q\zeta_1(u,q)\right], \\ &\psi_2(x) - \psi_2(u) - \zeta_2(u,q) + q^T\nabla_q\zeta_2(u,q) - F_{x,u}\left[\nabla_x\psi_2(u) + \nabla_q\zeta_2(u,q)\right]\} \notin K \end{aligned}$$

i.e., either

$$\psi_1(x) - \psi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q) - F_{x,u}\left[\nabla_x\psi_1(u) + \nabla_q\zeta_1(u,q)\right] \ngeqq 0$$

or

$$\psi_2(x) - \psi_2(u) - \zeta_2(u,q) + q^T\nabla_q\zeta_2(u,q) - F_{x,u}\left[\nabla_x\psi_2(u) + \nabla_q\zeta_2(u,q)\right] \ngeqq 0.$$

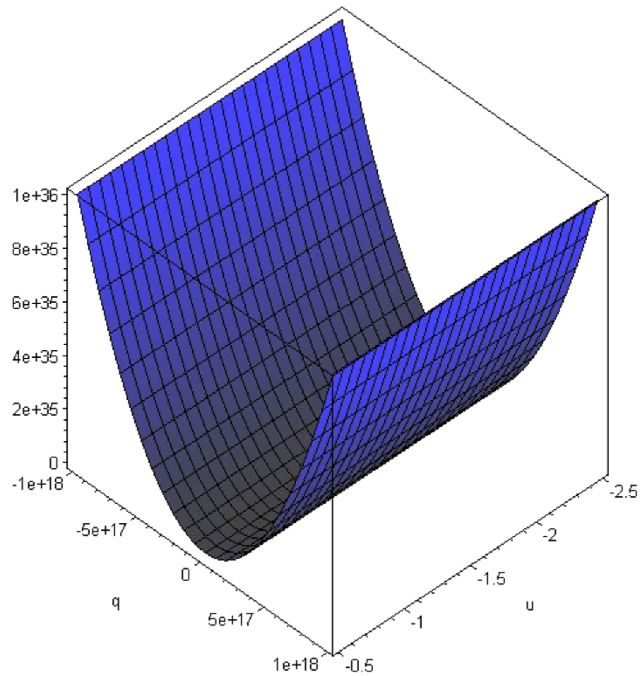

Figure 3 Graph of L_{22} (Expression (E3)) against q and u.

Since

$$\begin{aligned} M &= \psi_1(x) - \psi_1(u) - \zeta_1(u,q) + q^T\nabla_q\zeta_1(u,q) - F_{x,u}\big[\nabla_x\psi_1(u) + \nabla_q\zeta_1(u,q)\big] \\ &= x^3\sin\frac{2}{x} - u^3\sin\frac{2}{u} - 15u - 12(1-u)\left[-2u\cos\frac{2}{u} + 3u^2\sin\frac{2}{u} + 12\right] \\ &= -181.7330 < 0 \quad (\text{for } x = -2 \text{ and } u = -1). \end{aligned} \tag{E4}$$

In fact, $M < 0$ for all $x, u \in X$ as can be seen from Figure 4. Therefore, ψ is not higher-order K-F-convex with respect to ζ.

3 Wolfe type higher-order symmetric duality

In this section, we consider the following Wolfe type nondifferentiable multiobjective higher-order symmetric dual programs.

Primal problem (HNWP) K-minimize

$$\begin{aligned} G(x,y,\lambda,p) = f(x,y) &+ S(x \mid D)e_k + \big(\lambda^T h\big)(x,y,p)e_k - p^T\nabla_p\big(\lambda^T h\big)(x,y,p)e_k \\ &- y^T\nabla_y\big(\lambda^T f\big)(x,y)e_k - y^T\nabla_p\big(\lambda^T h\big)(x,y,p)e_k \end{aligned}$$

subject to

$$-\big\{\nabla_y\big(\lambda^T f\big)(x,y) - z + \nabla_p\big(\lambda^T h\big)(x,y,p)\big\} \in C_2^*, \tag{1}$$

$$z \in E, \tag{2}$$

$$\lambda^T e_k = 1, \tag{3}$$

$$\lambda \in \operatorname{int} K^*, \qquad x \in C_1. \tag{4}$$

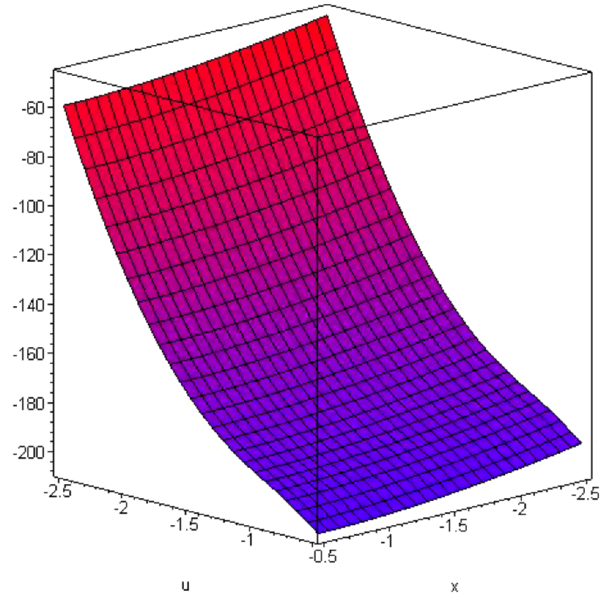

Figure 4 Graph of *M* (Expression (E4)) against *x* and *u*.

Dual problem (HNWD) K-maximize

$$H(u,v,\lambda,r) = f(u,v) - S(v \mid E)e_k + (\lambda^T g)(u,v,r)e_k - r^T \nabla_r (\lambda^T g)(u,v,r)e_k \\ - u^T \nabla_x (\lambda^T f)(u,v)e_k - u^T \nabla_r (\lambda^T g)(u,v,r)e_k$$

subject to

$$\nabla_x (\lambda^T f)(u,v) + w + \nabla_r (\lambda^T g)(u,v,r) \in C_1^*, \tag{5}$$

$$w \in D, \tag{6}$$

$$\lambda^T e_k = 1, \tag{7}$$

$$\lambda \in \operatorname{int} K^*, \qquad v \in C_2, \tag{8}$$

where

(i) C_1 and C_2 are closed convex cones with nonempty interiors in R^n and R^m, respectively,
(ii) $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$,
(iii) $f : S_1 \times S_2 \to R^k$, $h : S_1 \times S_2 \times R^m \to R^k$ and $g : S_1 \times S_2 \times R^n \to R^k$ are differentiable functions, $e_k = (1, \ldots, 1)^T \in R^k$, $\lambda \in R^k$,
(iv) r and p are vectors in R^n and R^m, respectively,
(v) D and E are compact convex sets in R^n and R^m, respectively, and
(vi) $S(x \mid D)$ and $S(v \mid E)$ are the support functions of D and E, respectively.

Remark 3 The problems (HNWP) and (HNWD) stated above are nondifferentiable because their objective function contains the support function $S(x \mid D)$ and $S(v \mid E)$.

We now prove the following duality results for the pair of problems (HNWP) and (HNWD).

Theorem 1 (Weak duality) *Let* (x,y,λ,z,p) *be feasible for the primal problem* (HNWP) *and* (u,v,λ,w,r) *be feasible for the dual problem* (HNWD). *Let the sublinear functionals* $F : R^n \times R^n \times R^n \mapsto R$ *and* $G : R^m \times R^m \times R^m \mapsto R$ *satisfy the following conditions*:

$$F_{x,u}(a) + \alpha_1^{-1} a^T u \geqq 0, \quad \textit{for all } a \in C_1^*, \tag{A}$$

$$G_{v,y}(b) + \alpha_2^{-1} b^T y \geqq 0, \quad \textit{for all } b \in C_2^*. \tag{B}$$

Suppose that (i) *either* $\sum_{i=1}^k \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2 + \rho_i^{(2)}(d_i^{(2)}(v,y))^2] \geqq 0$ *or* $\rho^{(1)} \geqq 0$ *and* $\rho^{(2)} \geqq 0$, (ii) $f(\cdot,v) + (\cdot)^T we_k$ *is higher-order* K-$(F,\alpha_1,\rho^{(1)},d^{(1)})$-*convex at* u *with respect to* $g(u,v,r)$, *and* (iii) $-\{f(x,\cdot) - (\cdot)^T ze_k\}$ *is higher-order* K-$(G,\alpha_2,\rho^{(2)},d^{(2)})$-*convex at* y *with respect to* $-h(x,y,p)$. *Then*

$$G(x,y,\lambda,p) - H(u,v,\lambda,r) \notin -K\setminus\{0\}. \tag{9}$$

Proof Since $f(\cdot,v) + (\cdot)^T we_k$ is higher-order K-$(F,\alpha_1,\rho^{(1)},d^{(1)})$-convex with respect to $g(u,v,r)$, we have

$$\begin{aligned}
&\big\{f_1(x,v) + x^T w - f_1(u,v) - u^T w - g_1(u,v,r) + r^T \nabla_r g_1(u,v,r) \\
&\quad - F_{x,u}\big[\alpha_1(x,u)\big(\nabla_x f_1(u,v) + w + \nabla_r g_1(u,v,r)\big)\big] - \rho_1^{(1)}\big(d_1^{(1)}(x,u)\big)^2, \ldots, \\
&\quad f_k(x,v) + x^T w - f_k(u,v) - u^T w - g_k(u,v,r) + r^T \nabla_r g_k(u,v,r) \\
&\quad - F_{x,u}\big[\alpha_1(x,u)\big(\nabla_x f_k(u,v) + w + \nabla_r g_k(u,v,r)\big)\big] - \rho_k^{(1)}\big(d_k^{(1)}(x,u)\big)^2\big\} \in K.
\end{aligned}$$

It follows from $\lambda \in \operatorname{int} K^*$, $\lambda^T e_k = 1$, and sublinearity of F that

$$\begin{aligned}
&\big(\lambda^T f\big)(x,v) + x^T w - \big(\lambda^T f\big)(u,v) - u^T w - \big(\lambda^T g\big)(u,v,r) \\
&\quad + r^T \nabla_r\big(\lambda^T g\big)(u,v,r) - F_{x,u}\big[\alpha_1(x,u)\big(\nabla_x\big(\lambda^T f\big)(u,v) + w + \nabla_r\big(\lambda^T g\big)(u,v,r)\big)\big] \\
&\quad - \sum_{i=1}^k \lambda_i \rho_i^{(1)}\big(d_i^{(1)}(x,u)\big)^2 \geqq 0.
\end{aligned} \tag{10}$$

As $-\{f(x,\cdot) - (\cdot)^T ze_k\}$ is higher-order K-$(G,\alpha_2,\rho^{(2)},d^{(2)})$-convex with respect to $-h(x,y,p)$, therefore we get

$$\begin{aligned}
&\big\{f_1(x,y) - y^T z - f_1(x,v) + v^T z + h_1(x,y,p) - p^T \nabla_p h_1(x,y,p) \\
&\quad - G_{v,y}\big[-\alpha_2(v,y)\big(\nabla_y f_1(x,y) - z + \nabla_p h_1(x,y,p)\big)\big] - \rho_1^{(2)}\big(d_1^{(2)}(v,y)\big)^2, \ldots, \\
&\quad f_k(x,y) - y^T z - f_k(x,v) + v^T z + h_k(x,y,p) - p^T \nabla_p h_k(x,y,p) \\
&\quad - G_{v,y}\big[-\alpha_2(v,y)\big(\nabla_y f_k(x,y) - z + \nabla_p h_k(x,y,p)\big)\big] - \rho_k^{(2)}\big(d_k^{(2)}(v,y)\big)^2\big\} \in K.
\end{aligned}$$

Again, using $\lambda \in \operatorname{int} K^*$, $\lambda^T e_k = 1$, and sublinearity of G, we obtain

$$\begin{aligned}
&\big(\lambda^T f\big)(x,y) - y^T z - \big(\lambda^T f\big)(x,v) + v^T z + \big(\lambda^T h\big)(x,y,p) - p^T \nabla_p\big(\lambda^T h\big)(x,y,p) \\
&\quad - G_{v,y}\big[-\alpha_2(v,y)\big(\nabla_y\big(\lambda^T f\big)(x,y) - z + \nabla_p\big(\lambda^T h\big)(x,y,p)\big)\big] \\
&\quad - \sum_{i=1}^k \lambda_i \rho_i^{(2)}\big(d_i^{(2)}(v,y)\big)^2 \geqq 0.
\end{aligned} \tag{11}$$

Further, adding the inequalities (10) and (11), we have

$$\begin{aligned}
&(\lambda^T f)(x,y) - (\lambda^T f)(u,v) + x^T w - u^T w - y^T z + v^T z + (\lambda^T h)(x,y,p) \\
&\quad - p^T \nabla_p (\lambda^T h)(x,y,p) - (\lambda^T g)(u,v,r) + r^T \nabla_r (\lambda^T g)(u,v,r) \\
&\geqq F_{x,u}\bigl[\alpha_1(x,u)\bigl(\nabla_x (\lambda^T f)(u,v) + w + \nabla_r (\lambda^T g)(u,v,r)\bigr)\bigr] \\
&\quad + G_{v,y}\bigl[-\alpha_2(v,y)\bigl(\nabla_y (\lambda^T f)(x,y) - z + \nabla_p (\lambda^T h)(x,y,p)\bigr)\bigr] \\
&\quad + \sum_{i=1}^{k} \lambda_i \bigl[\rho_i^{(1)}\bigl(d_i^{(1)}(x,u)\bigr)^2 + \rho_i^{(2)}\bigl(d_i^{(2)}(v,y)\bigr)^2\bigr].
\end{aligned} \tag{12}$$

Now, since (x,y,λ,z,p) is feasible for the primal problem (HNWP) and (u,v,λ,w,r) is feasible for the dual problem (HNWD), $\alpha_1(x,u) > 0$, by the dual constraint (5), the vector $a = \alpha_1(x,u)[\nabla_x(\lambda^T f)(u,v) + w + \nabla_r(\lambda^T g)(u,v,r)] \in C_1^*$, and so from the hypothesis (A), we obtain

$$F_{x,u}(a) + \alpha_1^{-1} a^T u \geqq 0. \tag{13}$$

Similarly,

$$G_{v,y}(b) + \alpha_2^{-1} b^T y \geqq 0, \tag{14}$$

for the vector $b = -\alpha_2(v,y)[\nabla_y(\lambda^T f)(x,y) - z + \nabla_p(\lambda^T h)(x,y,p)] \in C_2^*$.

Using (13), (14) and the hypothesis (i) in (12), we have

$$\begin{aligned}
&(\lambda^T f)(x,y) - (\lambda^T f)(u,v) + x^T w - u^T w - y^T z + v^T z + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) \\
&\quad - (\lambda^T g)(u,v,r) + r^T \nabla_r (\lambda^T g)(u,v,r) \geqq -\alpha_1^{-1} a^T u - \alpha_2^{-1} b^T y.
\end{aligned}$$

Further, substituting the values of a and b, we have

$$\begin{aligned}
&(\lambda^T f)(x,y) + x^T w + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) \\
&\quad - y^T \nabla_y (\lambda^T f)(x,y) - y^T \nabla_p (\lambda^T h)(x,y,p) \\
&\geqq (\lambda^T f)(u,v) - v^T z + (\lambda^T g)(u,v,r) - r^T \nabla_r (\lambda^T g)(u,v,r) \\
&\quad - u^T \nabla_x (\lambda^T f)(u,v) - u^T \nabla_r (\lambda^T g)(u,v,r).
\end{aligned}$$

In view of the fact that $x^T w \leqq S(x \mid D)$, $v^T z \leqq S(v \mid E)$ and $\lambda^T e_k = 1$, the last inequality yields

$$\begin{aligned}
&(\lambda^T f)(x,y) + S(x \mid D) + (\lambda^T h)(x,y,p) - p^T \nabla_p (\lambda^T h)(x,y,p) \\
&\quad - y^T \nabla_y (\lambda^T f)(x,y) - y^T \nabla_p (\lambda^T h)(x,y,p) \\
&\geqq (\lambda^T f)(u,v) - S(v \mid E) + (\lambda^T g)(u,v,r) - r^T \nabla_r (\lambda^T g)(u,v,r) \\
&\quad - u^T \nabla_x (\lambda^T f)(u,v) - u^T \nabla_r (\lambda^T g)(u,v,r).
\end{aligned} \tag{15}$$

Now, suppose contrary to the result that (9) does not hold, that is,

$$G(x,y,\lambda,p) - H(u,v,\lambda,r) \in -K \setminus \{0\},$$

or

$$\begin{aligned}&\{f(x,y)+S(x\mid D)e_k+(\lambda^T h)(x,y,p)e_k-p^T\nabla_p(\lambda^T h)(x,y,p)e_k-y^T\nabla_y(\lambda^T f)(x,y)e_k\\&\quad-y^T\nabla_p(\lambda^T h)(x,y,p)e_k\}-\{f(u,v)-S(v\mid E)e_k+(\lambda^T g)(u,v,r)e_k\\&\quad-r^T\nabla_r(\lambda^T g)(u,v,r)e_k-u^T\nabla_x(\lambda^T f)(u,v)e_k-u^T\nabla_r(\lambda^T g)(u,v,r)e_k\}\in -K\setminus\{0\}.\end{aligned}$$

It follows from $\lambda\in \operatorname{int}K^*$ and $\lambda^T e_k=1$ that

$$\begin{aligned}&(\lambda^T f)(x,y)+S(x\mid D)+(\lambda^T h)(x,y,p)-p^T\nabla_p(\lambda^T h)(x,y,p)\\&\quad-y^T\nabla_y(\lambda^T f)(x,y)-y^T\nabla_p(\lambda^T h)(x,y,p)\\&<(\lambda^T f)(u,v)-S(v\mid E)+(\lambda^T g)(u,v,r)-r^T\nabla_r(\lambda^T g)(u,v,r)\\&\quad-u^T\nabla_x(\lambda^T f)(u,v)-u^T\nabla_r(\lambda^T g)(u,v,r),\end{aligned}$$

which contradicts (15). Hence the result. □

Theorem 2 (Strong duality) *Let $f:S_1\times S_2\to R^k$ be a twice differentiable function and let $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ be a weak efficient solution of* (HNWP). *Suppose that*

(i) *the matrix $\nabla_{pp}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})$ is nonsingular,*
(ii) *the vectors $\{\nabla_y f_1(\bar{x},\bar{y}),\dots,\nabla_y f_k(\bar{x},\bar{y})\}$ are linearly independent,*
(iii) *the vector $\{\nabla_y(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})-\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})+\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}\}\notin$ $\operatorname{span}\{\nabla_y f_1(\bar{x},\bar{y}),\dots,\nabla_y f_k(\bar{x},\bar{y})\}\setminus\{0\}$,*
(iv) *$\nabla_y(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})-\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})+\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}=0$ implies $\bar{p}=0$,*
(v) *$(\bar{\lambda}^T h)(\bar{x},\bar{y},0)=(\bar{\lambda}^T g)(\bar{x},\bar{y},0)$, $\nabla_y(\bar{\lambda}^T h)(\bar{x},\bar{y},0)=0$, $\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},0)=0$, $\nabla_x(\bar{\lambda}^T h)(\bar{x},\bar{y},0)=\nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},0)$ and*
(vi) *K is a closed convex pointed cone with $R^k_+\subseteq K$.*

Then

(I) *there exists $\bar{w}\in D$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r}=0)$ is feasible for* (HNWD), *and*
(II) *$G(\bar{x},\bar{y},\bar{\lambda},\bar{p})=H(\bar{x},\bar{y},\bar{\lambda},\bar{r})$.*

Also, if the hypotheses of Theorem 1 *are satisfied for all feasible solutions of* (HNWP) *and* (HNWD), *then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r}=0)$ is an efficient solution for* (HNWD).

Proof Since $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a weak efficient solution of (HNWP), by the Fritz John necessary optimality conditions [33], there exist $\bar{\alpha}\in K^*$, $\bar{\beta}\in C_2$, $\bar{\eta}\in R$, such that the following conditions are satisfied at $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$:

$$\begin{aligned}&\{\bar{\alpha}^T(\nabla_x f(\bar{x},\bar{y})+\bar{\gamma}e_k)+\nabla_x(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})(\bar{\alpha}^T e_k)\\&\quad+\nabla_{xy}(\bar{\lambda}^T f)(\bar{x},\bar{y})[\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}]+\nabla_{px}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})\\&\quad\times[\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\bar{p})]\}(x-\bar{x})\geqq 0,\quad \text{for all } x\in C_1,\end{aligned}\tag{16}$$

$$\begin{aligned}&\nabla_y f(\bar{x},\bar{y})[\bar{\alpha}-(\bar{\alpha}^T e_k)\bar{\lambda}]+[\nabla_y(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})\\&\quad-\nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})](\bar{\alpha}^T e_k)+\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})[\bar{\beta}-(\bar{\alpha}^T e_k)\bar{y}]\\&\quad+\nabla_{py}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})[\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\bar{p})]=0,\end{aligned}\tag{17}$$

$$\nabla_{pp}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})[\bar{\beta}-(\bar{\alpha}^T e_k)(\bar{y}+\bar{p})]=0,\tag{18}$$

$$\begin{aligned}&\big\{\nabla_y f(\bar{x},\bar{y})\big[\bar{\beta}-\big(\bar{\alpha}^T e_k\big)\bar{y}\big]+h(\bar{x},\bar{y},\bar{p})\big(\bar{\alpha}^T e_k\big)+\bar{\eta}e_k\\&\quad+\nabla_p h(\bar{x},\bar{y},\bar{p})\big[\bar{\beta}-\big(\bar{\alpha}^T e_k\big)(\bar{y}+\bar{p})\big]\big\}(\lambda-\bar{\lambda})\geqq 0,\quad \text{for all } \lambda\in \operatorname{int}K^*,\end{aligned}\tag{19}$$

$$\bar{\beta}^T\big[\nabla_y\big(\bar{\lambda}^T f\big)(\bar{x},\bar{y})-\bar{z}+\nabla_p\big(\bar{\lambda}^T h\big)(\bar{x},\bar{y},\bar{p})\big]=0,\tag{20}$$

$$\bar{\eta}^T\big[\bar{\lambda}^T e_k-1\big]=0,\tag{21}$$

$$\bar{\beta}\in N_E(\bar{z}),\tag{22}$$

$$\bar{\gamma}\in D,\quad \bar{\gamma}^T\bar{x}=S(\bar{x}\mid D),\tag{23}$$

$$(\bar{\alpha},\bar{\beta},\bar{\eta})\neq 0.\tag{24}$$

From (18) and nonsingularity of $\nabla_{pp}(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})$, we have

$$\bar{\beta}=\big(\bar{\alpha}^T e_k\big)(\bar{y}+\bar{p}).\tag{25}$$

Also, (19) is equivalent to

$$\nabla_y f(\bar{x},\bar{y})\big[\bar{\beta}-\big(\bar{\alpha}^T e_k\big)\bar{y}\big]+h(\bar{x},\bar{y},\bar{p})\big(\bar{\alpha}^T e_k\big)+\bar{\eta}e_k+\nabla_p h(\bar{x},\bar{y},\bar{p})\big[\bar{\beta}-\big(\bar{\alpha}^T e_k\big)(\bar{y}+\bar{p})\big]=0.$$

If $\bar{\alpha}=0$, then (25) yields $\bar{\beta}=0$. Further, the above equality gives $\bar{\eta}e_k=0$ or $\bar{\eta}=0$. Consequently, $(\bar{\alpha},\bar{\beta},\bar{\eta})=0$, contradicting (24). Hence, $\bar{\alpha}\neq 0$.

Since $R_+^k\subseteq K\Rightarrow K^*\subseteq R_+^k$, therefore $\bar{\alpha}\in K^*\Rightarrow\bar{\alpha}\geqq 0$.

But $\bar{\alpha}\neq 0\Rightarrow\bar{\alpha}\geq 0$ or

$$\bar{\alpha}^T e_k>0.\tag{26}$$

Now, using (25) and (26) in (17), we get

$$\begin{aligned}&\nabla_y\big(\bar{\lambda}^T h\big)(\bar{x},\bar{y},\bar{p})-\nabla_p\big(\bar{\lambda}^T h\big)(\bar{x},\bar{y},\bar{p})+\nabla_{yy}\big(\bar{\lambda}^T f\big)(\bar{x},\bar{y})\bar{p}\\&\quad=-\frac{1}{\bar{\alpha}^T e_k}\nabla_y f(\bar{x},\bar{y})\big[\bar{\alpha}-\big(\bar{\alpha}^T e_k\big)\bar{\lambda}\big],\end{aligned}\tag{27}$$

which, by the hypotheses (iii) and (iv), implies

$$\bar{p}=0.\tag{28}$$

Using the hypothesis (iii) in (27), we obtain

$$\nabla_y f(\bar{x},\bar{y})\big[\bar{\alpha}-\big(\bar{\alpha}^T e_k\big)\bar{\lambda}\big]=0.$$

Since the vectors $\{\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})\}$ are linearly independent, therefore the above equation yields

$$\bar{\alpha}=\big(\bar{\alpha}^T e_k\big)\bar{\lambda}.\tag{29}$$

From (28) in (25), we get

$$\bar{\beta}=\big(\bar{\alpha}^T e_k\big)\bar{y}.\tag{30}$$

Using (26) and (28)-(30) in (16), we have

$$\left\{\nabla_x\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y}) + \bar{\gamma} + \nabla_x\left(\bar{\lambda}^T h\right)(\bar{x},\bar{y},\bar{p})\right\}(x-\bar{x}) \geqq 0, \quad \text{for all } x \in C_1.$$

From the hypothesis (v), for $\bar{r}=0$, the above inequality yields

$$\left\{\nabla_x\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y}) + \bar{\gamma} + \nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})\right\}(x-\bar{x}) \geqq 0. \tag{31}$$

Let $x \in C_1$. Then $x+\bar{x} \in C_1$, and so (31) implies

$$\left\{\nabla_x\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y}) + \bar{\gamma} + \nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})\right\}^T x \geqq 0, \quad \text{for all } x \in C_1.$$

Therefore,

$$\left\{\nabla_x\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y}) + \bar{\gamma} + \nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})\right\} \in C_1^*. \tag{32}$$

Also, from (26), (30) and $\bar{\beta} \in C_2$, we obtain $\bar{y} \in C_2$. Thus $(\bar{x},\bar{y},\bar{\lambda},\bar{w}=\bar{\gamma},\bar{r}=0)$, satisfies the dual constraints from (5) to (8) in (HNWD), and so it is a feasible solution for the dual problem (HNWD). Now, letting $x=0$ and $x=2\bar{x}$ in (31), we get

$$\begin{aligned}
&\bar{x}^T\left[\nabla_x\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y}) + \bar{\gamma} + \nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})\right] = 0, \quad \text{or} \\
&\bar{x}^T\left[\nabla_x\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y}) + \nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})\right] = -\bar{x}^T\bar{\gamma} = -S(\bar{x}\mid D).
\end{aligned} \tag{33}$$

From (22) and (30), $(\bar{\alpha}^T e_k)\bar{y} \in N_E(\bar{z})$. Since $\bar{\alpha}^T e_k > 0$, $\bar{y} \in N_E(\bar{z})$. Also, as E is a compact convex set in R^m, $\bar{y}^T\bar{z} = S(\bar{y}\mid E)$.

Further, from (20), (26) and (30) and the above relation, we obtain

$$\bar{y}^T\left[\nabla_y\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y}) + \nabla_p\left(\bar{\lambda}^T h\right)(\bar{x},\bar{y},\bar{p})\right] = \bar{y}^T\bar{z} = S(\bar{y}\mid E). \tag{34}$$

Therefore, using (28), (33), (34) and the hypothesis (v), for $\bar{r}=0$, we get

$$\begin{aligned}
&f(\bar{x},\bar{y}) + S(\bar{x}\mid D)e_k + \left(\bar{\lambda}^T h\right)(\bar{x},\bar{y},\bar{p})e_k - \bar{p}^T\nabla_p\left(\bar{\lambda}^T h\right)(\bar{x},\bar{y},\bar{p})e_k \\
&\quad - \bar{y}^T\nabla_y\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y})e_k - \bar{y}^T\nabla_p\left(\bar{\lambda}^T h\right)(\bar{x},\bar{y},\bar{p})e_k \\
&= f(\bar{x},\bar{y}) - S(\bar{y}\mid E)e_k + \left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})e_k - \bar{r}^T\nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})e_k \\
&\quad - \bar{x}^T\nabla_x\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y})e_k - \bar{x}^T\nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})e_k,
\end{aligned}$$

that is, the two objective values are equal.

Now, let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r}=0)$ be not an efficient solution of (HNWD), then there exists $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{r}=0)$ feasible for (HNWD) such that

$$\begin{aligned}
&f(\bar{x},\bar{y}) - S(\bar{y}\mid E)e_k + \left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})e_k - \bar{r}^T\nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})e_k - \bar{x}^T\nabla_x\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y})e_k \\
&\quad - \bar{x}^T\nabla_r\left(\bar{\lambda}^T g\right)(\bar{x},\bar{y},\bar{r})e_k - f(\bar{u},\bar{v}) + S(\bar{v}\mid E)e_k - \left(\bar{\lambda}^T g\right)(\bar{u},\bar{v},\bar{r})e_k + \bar{r}^T\nabla_r\left(\bar{\lambda}^T g\right)(\bar{u},\bar{v},\bar{r})e_k \\
&\quad + \bar{u}^T\nabla_x\left(\bar{\lambda}^T f\right)(\bar{u},\bar{v})e_k + \bar{u}^T\nabla_r\left(\bar{\lambda}^T g\right)(\bar{u},\bar{v},\bar{r})e_k \in -K\setminus\{0\}.
\end{aligned}$$

As $\bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_r(\bar{\lambda}^T g)(\bar{x},\bar{y},\bar{r})] = -S(\bar{x} \mid D)$, $\bar{y}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_p(\bar{\lambda}^T h)(\bar{x},\bar{y},\bar{p})] = S(\bar{y} \mid E)$ and from the hypothesis (v), for $\bar{r} = 0$, we obtain

$$\begin{aligned}&\bigl\{f(\bar{x},\bar{y}) + S(\bar{x} \mid D)e_k + \bigl(\bar{\lambda}^T h\bigr)(\bar{x},\bar{y},\bar{p})e_k - \bar{p}^T\nabla_p\bigl(\bar{\lambda}^T h\bigr)(\bar{x},\bar{y},\bar{p})e_k - \bar{y}^T\nabla_y\bigl(\bar{\lambda}^T f\bigr)(\bar{x},\bar{y})e_k \\ &\quad - \bar{y}^T\nabla_p\bigl(\bar{\lambda}^T h\bigr)(\bar{x},\bar{y},\bar{p})e_k\bigr\} - \bigl\{f(\bar{u},\bar{v}) - S(\bar{v} \mid E)e_k + \bigl(\bar{\lambda}^T g\bigr)(\bar{u},\bar{v},\bar{r})e_k \\ &\quad - \bar{r}^T\nabla_r\bigl(\bar{\lambda}^T g\bigr)(\bar{u},\bar{v},\bar{r})e_k - \bar{u}^T\nabla_x\bigl(\bar{\lambda}^T f\bigr)(\bar{u},\bar{v})e_k - \bar{u}^T\nabla_r\bigl(\bar{\lambda}^T g\bigr)(\bar{u},\bar{v},\bar{r})e_k\bigr\} \in -K\backslash\{0\},\end{aligned}$$

which contradicts Theorem 1. Hence, $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{r} = 0)$ is an efficient solution of (HNWD). □

Theorem 3 (Converse duality) *Let $f : S_1 \times S_2 \to R^k$ be a twice differentiable function and let $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{r})$ be a weak efficient solution of* (HNWD). *Suppose that*

(i) *the matrix $\nabla_{rr}(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r})$ is nonsingular,*
(ii) *the vectors $\{\nabla_x f_1(\bar{u},\bar{v}),\ldots,\nabla_x f_k(\bar{u},\bar{v})\}$ are linearly independent,*
(iii) *the vector $\{\nabla_x(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r}) - \nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r}) + \nabla_{xx}(\bar{\lambda}^T f)(\bar{u},\bar{v})\bar{r}\} \notin \operatorname{span}\{\nabla_x f_1(\bar{u},\bar{v}),\ldots,\nabla_x f_k(\bar{u},\bar{v})\}\backslash\{0\}$,*
(iv) *$\nabla_x(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r}) - \nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},\bar{r}) + \nabla_{xx}(\bar{\lambda}^T f)(\bar{u},\bar{v})\bar{r} = 0$ implies $\bar{r} = 0$,*
(v) *$(\bar{\lambda}^T g)(\bar{u},\bar{v},0) = (\bar{\lambda}^T h)(\bar{u},\bar{v},0)$, $\nabla_x(\bar{\lambda}^T g)(\bar{u},\bar{v},0) = 0$, $\nabla_r(\bar{\lambda}^T g)(\bar{u},\bar{v},0) = 0$, $\nabla_y(\bar{\lambda}^T g)(\bar{u},\bar{v},0) = \nabla_p(\bar{\lambda}^T h)(\bar{u},\bar{v},0)$ and*
(vi) *K is a closed convex pointed cone with $R^k_+ \subseteq K$.*

Then

(I) *there exists $\bar{z} \in E$ such that $(\bar{u},\bar{v},\bar{\lambda},\bar{z},\bar{p} = 0)$ is feasible for* (HNWP), *and*
(II) *$G(\bar{u},\bar{v},\bar{\lambda},\bar{p}) = H(\bar{u},\bar{v},\bar{\lambda},\bar{r})$.*

Also, if the hypotheses of Theorem 1 *are satisfied for all feasible solutions of* (HNWP) *and* (HNWD), *then $(\bar{u},\bar{v},\bar{\lambda},\bar{z},\bar{p} = 0)$ is an efficient solution for* (HNWP).

Proof It follows on the lines of Theorem 2. □

4 Special cases

In this section, we consider some of the special cases of the problems studied in Section 3. In all these cases, $K = R^k_+$, $C_1 = R^n_+$, $C_2 = R^m_+$, $(\lambda^T h)(x,y,p) = \frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p$ and $(\lambda^T g)(u,v,r) = \frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r$.

(i) For $k = 1$, $D = \{Ay : y^T Ay \leqq 1\}$, $E = \{Bx : x^T Bx \leqq 1\}$, where A and B are positive semidefinite matrices, $(x^T Ax)^{1/2} = S(x \mid D)$ and $(y^T By)^{1/2} = S(y \mid E)$. In this case, (HNWP) and (HNWD) reduce to the problems considered in Ahmad and Husain [4].
(ii) Let $k = 1$, $D = \{0\}$ and $E = \{0\}$, then our problems (HNWP) and (HNWD) become the problems studied in Gulati *et al.* [8].
(iii) If $k = 1$, then our problems (HNWP) and (HNWD) reduce to the programs studied in Yang *et al.* [16].
(iv) Let $D = \{0\}$ and $E = \{0\}$, our problems reduce to (MP) and (MD) considered in Yang *et al.* [17] along with nonnegativity restrictions $x \geqq 0$ and $v \geqq 0$. However, taking $F_{x,u}(a) = (x-u)^T a$ and $G_{v,y}(b) = (v-y)^T b$ along with the hypotheses (A) and (B) of Theorem 1 in [17] gives $x \geqq 0$ and $v \geqq 0$.

5 Conclusions

A pair of Wolfe-type multiobjective higher-order symmetric dual programs involving nondifferentiable functions over arbitrary cones has been formulated. Further, an example of higher-order K-(F, α, ρ, d)-convex which is not higher-order K-F-convex has been illustrated. Weak, strong and converse duality theorems under higher-order K-(F, α, ρ, d)-convexity assumptions have also been established. It is to be noted that some of the known results, including those of Ahmad and Husain [4], Gulati *et al.* [8] and Yang *et al.* [16, 17], are special cases of our study. This work can be further extended to study nondifferentiable higher-order multiobjective symmetric dual programs over arbitrary cones with different p_i's and different support functions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Indian Institute of Technology, Patna, 800 013, India. [2]Department of Applied Sciences, Gulzar Institute of Engineering and Technology, Khanna, 141 401, India. [3]Faculty, Operations & Information Systems, Indian Institute of Management, Udaipur, 313 001, India.

Acknowledgements
The authors wish to thank the referee for her/his valuable suggestions which have considerably improved the presentation of the paper.

Received: 24 August 2012 Accepted: 10 November 2012 Published: 14 December 2012

References

1. Mangasarian, OL: Second and higher-order duality in nonlinear programming. J. Math. Anal. Appl. **51**, 607-620 (1975)
2. Agarwal, RP, Ahmad, I, Jayswal, A: Higher-order symmetric duality in nondifferentiable multiobjective programming problems involving generalized cone convex functions. Math. Comput. Model. **52**, 1644-1650 (2010)
3. Ahmad, I: Second-order symmetric duality in nondifferentiable multiobjective programming. Inf. Sci. **173**, 23-34 (2005)
4. Ahmad, I, Husain, Z: On nondifferentiable second-order symmetric duality in mathematical programming. Indian J. Pure Appl. Math. **35**, 665-676 (2004)
5. Ahmad, I, Husain, Z: On multiobjective second-order symmetric duality with cone constraints. Eur. J. Oper. Res. **204**, 402-409 (2010)
6. Ahmad, I, Husain, Z, Sharma, S: Higher-order duality in nondifferentiable multiobjective programming. Numer. Funct. Anal. Optim. **28**, 989-1002 (2007)
7. Chen, X: Higher-order symmetric duality in nondifferentiable multiobjective programming problems. J. Math. Anal. Appl. **290**, 423-435 (2004)
8. Gulati, TR, Ahmad, I, Husain, I: Second-order symmetric duality with generalized convexity. Opsearch **38**, 210-222 (2001)
9. Gulati, TR, Gupta, SK: Higher-order nondifferentiable symmetric duality with generalized *F*-convexity. J. Math. Anal. Appl. **329**, 229-237 (2007)
10. Gulati, TR, Gupta, SK: Higher-order symmetric duality with cone constraints. Appl. Math. Lett. **22**, 776-781 (2009)
11. Gulati, TR, Saini, H, Gupta, SK: Second-order multiobjective symmetric duality with cone constraints. Eur. J. Oper. Res. **205**, 247-252 (2010)
12. Gupta, SK, Kailey, N: A note on multiobjective second-order symmetric duality. Eur. J. Oper. Res. **201**, 649-651 (2010)
13. Hanson, MA: Second-order invexity and duality in mathematical programming. Opsearch **30**, 313-320 (1993)
14. Srivastava, MK, Bhatia, M: Symmetric duality for multiobjective programming using second-order (F, ρ)-convexity. Opsearch **43**, 274-295 (2006)
15. Suneja, SK, Lalitha, CS, Khurana, S: Second-order symmetric duality in multiobjective programming. Eur. J. Oper. Res. **144**, 492-500 (2003)
16. Yang, XM, Yang, XQ, Teo, KL: Non-differentiable second-order symmetric duality in mathematical programming with *F*-convexity. Eur. J. Oper. Res. **144**, 554-559 (2003)
17. Yang, XM, Yang, XQ, Teo, KL, Hou, SH: Multiobjective second-order symmetric duality with *F*-convexity. Eur. J. Oper. Res. **165**, 585-591 (2005)
18. Yang, XM, Yang, XQ, Teo, KL, Hou, SH: Second order duality for nonlinear programming. Indian J. Pure Appl. Math. **35**, 699-708 (2004)
19. Zhang, J, Mond, B: Second order duality for multiobjective nonlinear programming involving generalized convexity. In: Glower, BM, Craven, BD, Ralph, D (eds.) Proceedings of the Optimization Miniconference III, pp. 79-95. University of Ballarat, Ballarat (1997)
20. Miettinen, KM: Nonlinear Multiobjective Optimization. Kluwer, Boston (1999)

21. Pardalos, PM, Siskos, Y, Zopounidis, C: Advances in Multicriteria Analysis. Kluwer, Netherlands (1995)
22. Chinchuluun, A, Pardalos, PM: A survey of recent developments in multiobjective optimization. Ann. Oper. Res. **154**, 29-50 (2007)
23. Gulati, TR, Geeta, M: On some symmetric dual models in multiobjective programming. Appl. Math. Comput. **215**, 380-383 (2009)
24. Preda, V: On efficiency and duality for multiobjective programs. J. Math. Anal. Appl. **166**, 356-377 (1992)
25. Hanson, MA, Mond, B: Further generalization of convexity in mathematical programming. J. Inf. Optim. Sci. **3**, 25-32 (1986)
26. Vial, JP: Strong and weak convexity of sets and functions. Math. Oper. Res. **8**, 231-259 (1983)
27. Liang, ZA, Huang, HX, Pardalos, PM: Optimality conditions and duality for a class of nonlinear fractional programming problems. J. Optim. Theory Appl. **110**, 611-619 (2001)
28. Liang, ZA, Huang, HX, Pardalos, PM: Efficiency conditions and duality for a class of multiobjective fractional programming problems. J. Glob. Optim. **27**, 447-471 (2003)
29. Ahmad, I, Husain, Z: Second order (F, α, ρ, d)-convexity and duality in multiobjective programming. Inf. Sci. **176**, 3094-3103 (2006)
30. Ahmad, I, Husain, Z: Second order symmetric duality in multiobjective programming involving cones. Optim. Lett. (2012, article in press). doi:10.1007/s11590-012-0508-2
31. Mishra, SK, Lai, KK: Second-order symmetric duality in multiobjective programming involving generalized cone-invex functions. Eur. J. Oper. Res. **178**, 20-26 (2007)
32. Agarwal, RP, Ahmad, I, Gupta, SK: A note on higher-order nondifferentiable symmetric duality in multiobjective programming. Appl. Math. Lett. **24**, 1308-1311 (2011)
33. Suneja, SK, Aggarwal, S, Davar, S: Multiobjective symmetric duality involving cones. Eur. J. Oper. Res. **141**, 471-479 (2002)
34. Khurana, S: Symmetric duality in multiobjective programming involving generalized cone-invex functions. Eur. J. Oper. Res. **165**, 592-597 (2005)

doi:10.1186/1029-242X-2012-298
Cite this article as: Gupta et al.: **Duality for nondifferentiable multiobjective higher-order symmetric programs over cones involving generalized (F, α, ρ, d)-convexity.** *Journal of Inequalities and Applications* 2012 **2012**:298.